Geometry of variational methods:
Generalized group theoretic coherent states

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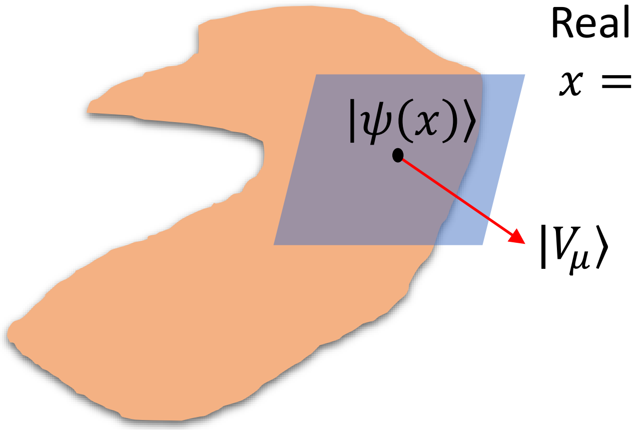
Abstract: Group theoretic coherent states (Gilmore-Perelomov) provide a mathematical formalism to understand a classify a large number of prominent variational families, including regular coherent states, spin-coherent states and bosonic-fermionic Gaussian states. In this work, we extend such these families by inducing entanglement through the action of operators that are quadratic in Cartan subalgebra elements.
The resulting families are promising candidates to study approximate ground states and quenches in a large class of condensed matter systems.

1. Geometry of variational family

Variational family

Manifold $\mathcal{M}$



Real parameters
$x = (x_1, \ldots, x_n)$

Tangent vectors form <u>real</u> vector space:
$\mathcal{T}_\psi\mathcal{M} = \mathrm{span}_{\mathbb{R}}(|V_\mu\rangle)$

Kähler structures $\langle V_\mu|V_\nu\rangle = \frac{1}{2}(g_{\mu\nu} + i\,\omega_{\mu\nu})$

Metric $G^{\mu\nu} = (g^{-1})^{\mu\nu}$
Symplectic form $\Omega^{\mu\nu} = (\omega^{-1})^{\mu\nu}$

Complex structure
$J^\mu{}_\nu = G^{\mu\sigma}\omega_{\sigma\nu}$

$\mathcal{M}$ is Kähler $\Leftrightarrow J^2 = -\mathbb{1}$

Real time evolution $\mathcal{X}^\mu = -\Omega^{\mu\nu}(\partial_\mu E)$

Imaginary time evolution $\mathcal{F}^\mu = G^{\mu\nu}(\partial_\mu E)$

2. Group theoretic coherent states

Lie group $g \in \mathcal{G}$ Lie algebra $\Xi_i \in \mathfrak{g}$ Hilbert space $\mathcal{H}$

Representation $U(g): \mathcal{H} \to \mathcal{H}$ with $U(g)U(h) = U(gh)$

Construction of group theoretic coherent states

Step 1: Choose Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$

$[\hat{H}_I, \hat{H}_J] = 0$ $[\hat{H}_I, \hat{E}_\alpha] = \alpha_I \hat{E}_\alpha$ roots α_I

$\hat{E}_\alpha^\dagger = \hat{E}_{-\alpha}$

Step 2: Find highest weight vector $|\phi\rangle \in \mathcal{H}$

$\hat{E}_\alpha|\phi\rangle = 0$ for all $\alpha > 0$ (positive roots)

Step 3: Family of group theoretic coherent states

$$|\psi(g)\rangle = U(g)|\phi\rangle$$

Example 1: SU(2) spin coherent states

$$|\psi\rangle = U|\downarrow, \dots, \downarrow\rangle$$

$U = \exp(c_s^i \hat{\sigma}_i^s)$ with $\sigma_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$

Example 2: Bosonic Gaussian states

$$|\psi\rangle = U|0\rangle = \sqrt{\det(1 - \gamma\gamma^\dagger)}\, e^{i\,\gamma^{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger}|0\rangle$$

$U = \exp(i\,h_{ab}\hat{\xi}^a\hat{\xi}^b)$ with $\hat{\xi}^a = (\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_N, \hat{a}_N^\dagger)$

Example 3: Odd bosonic Gaussian states

$$|\psi\rangle = U \underbrace{|1,0,\dots,0\rangle}_{\text{single excitation}}$$

$U = \exp(i\,h_{ab}\hat{\xi}^a\hat{\xi}^b)$ with $\hat{\xi}^a = (\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_N, \hat{a}_N^\dagger)$

(All group theoretic coherent states form Kähler manifolds.)

3. Generalized group theoretic coherent states

Generalized group theoretic coherent state

$$|W,g,h\rangle = \underbrace{U(g)\, e^{i\, W^{IJ}\hat{H}_I\hat{H}_J}}_{\text{generalized}}\ \underbrace{U(h)\,|\phi\rangle}_{\text{coherent}}$$

for Cartan subalgebra $H_I \in \mathfrak{h}$

Efficient evaluation of expectation values

Expectation value $\langle W,g,h|E_{\alpha_1} \dots E_{\alpha_m}|W,g,h\rangle$

Transformation $U_W = e^{i\, W^{IJ}\hat{H}_I\hat{H}_J}$ leads to

$$U_W^\dagger \hat{E}_\alpha U_W = e^{i\, W^{IJ}(\hat{H}_I\alpha_J + \hat{H}_J\alpha_I)}\hat{E}_\alpha$$

which can be treated with standard group theoretic calculation techniques.

Example 1: Generalized SU(2) spin coherent states

$$|W,g,h\rangle = U(g)\, e^{i\, W^{ij}\hat{\sigma}_i^z\hat{\sigma}_j^z}\, U(h)|0\rangle$$

$U(g) = \exp(c_s^i \hat{\sigma}_i^s)$ with $\sigma_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$

Example 2: Generalized Gaussian states

$$|W,g,h\rangle = U(g)\, e^{i\, W^{ij}\hat{n}_i\hat{n}_j}\, U(h)|0\rangle$$

$U(g) = \exp(i\, h_{ab}\hat{\xi}^a\hat{\xi}^b)$ with $\hat{\xi}^a = (\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_N, \hat{a}_N^\dagger)$

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